Journal of Sound and Vibration (1999) **222**(3), 409–423
Article No. jsvi.1998.2083, available online at http://www.idealibrary.com on IDEAL

NATURAL FREQUENCIES OF A BEAM WITH AN ARBITRARY NUMBER OF CRACKS

E. I. SHIFRIN
Department of Mathematics, Moscow State Aviation Technology University, Orshanskaya str. 3, Moscow, Russia

AND

R. RUOTOLO
Department of Aeronautical and Space Engineering, Politecnico di Torino, corso Duca degli Abruzzi 24, Torino, Italy

(*Received 2 April* 1998, *and in final form 9 November* 1998)

In this article a new technique is proposed for calculating natural frequencies of a vibrating beam with an arbitrary finite number of transverse open cracks. The main feature of this method is related to decreasing the dimension of the matrix involved in the calculation, so that reduced computation time is required for evaluating natural frequencies compared to alternative methods which also make use of a continuous model of the beam.

1. INTRODUCTION

During the last decade, the use of vibration based inspection has been a topic of active research. Several authors have proposed techniques to estimate the effects of damage on the eigenparameters on the structure under study (usually known as the direct problem), while others have dealt with the problem of detecting, locating and quantifying the extent of damage (known as the inverse problem). An extended literature review of these methods can be found in reference [1], while Dimarogonas [2] presents a state of the art review of methods developed to deal with cracked structures.

In order to investigate the prevailing effects of damage present in the structure under examination, several studies introduced damage into the mathematical model through a simple reduction of the stiffness on a given zone of the structure [1–4].

To investigate the variation in dynamic properties due to the presence of real damage, in several papers the evaluation of changes in natural frequencies of a simple cantilever beam due to the presence of one or, at most, two cracks is addressed [5–15]. Moreover, in the same papers, the method for calculating the shift in natural frequencies is also often proposed in order to address the inverse problem.

0022–460X/99/180409 + 15 $30.00/0

The evaluation of changes in eigenfrequencies due to the presence of cracks, notches and other geometrical discontinuities was developed by Gudmunson [16] using a theory based on a first-order perturbation. Christides and Barr [17] developed a cracked Euler–Bernoulli beam theory by deriving the differential equation and related boundary conditions for a uniform beam with one or more pairs of symmetric cracks.

To deal with the effects of cracks on the eigenparameters, in some articles the beam was subdivided into several beams, separated from one another by a crack, which was represented through a massless rotational spring [5–12]. Usually, the well-known general solution for the eigenfunctions of every beam in flexural motion was used, such that a linear system of algebraic equations was obtained by applying both end conditions and conditions related to the presence of the massless springs. Finally, it was possible to evaluate natural frequencies simply by finding roots of the determinant of the coefficient matrix of the linear system. Clearly a determinant search is time consuming, but if a continuous model is involved in the solution procedure a determinant search will be the best method of choice.

The general solution for the eigenfunctions of every beam contains four unknown constants. In this way the extension of the approach developed in reference [5] leads to a system of $(4n + 4)$ equations in the case of n cracks. Anyway, it should be noted that to construct the linear system by using the method proposed in reference [5] for a general case of n cracks is not a simple task. This is the main reason for which cases of just one crack [5] and two cracks [6] were considered in detail, without attempting to provide a solution for a more general situation. Furthermore, an approximate asymptotical approach for calculating eigenfrequencies was proposed in references [9–12].

Another way for evaluating natural frequencies of a cracked beam is based on the use of the finite element method [13–20]. By using this approximation technique it is possible to evaluate the dynamic properties of a cracked beam for an arbitrary number of cracks; also this method leads to a system of linear equations and to determinants with high order. Moreover, it should be recalled that theoretically a finite element model of a beam is less accurate than a continuous model.

In this paper a new method for evaluating natural frequencies of a beam with an arbitrary number of cracks is presented. This method is based on the use of massless rotational springs in order to represent the cracks and, as a main feature, leads to a system of $(n+2)$ linear equations for a beam with n cracks. As a consequence, due to the decrease in the determinant order in comparison with previously developed procedures, it is possible to reduce considerably the computer time needed to calculate the natural frequencies. This aspect is of a great importance from the inverse problem point of view, particularly if damage is located and quantified by using optimisation techniques, as in references [21–26].

2. CALCULATION OF THE NATURAL FREQUENCIES

A beam with length l and with n cracks is considered. It is assumed that the cracks are located at points x_1, $x_2 \ldots x_n$ such that $0 < x_1 < x_2 < \ldots < x_n < l$. Amplitudes of transverse displacement of the beam axis under time-harmonic vibration are denoted by $y_j(x)$ on the interval $x_{j-1} < x < x_j$, where $j = 1, 2, \ldots, n+1$, $x_0 = 0$ and $x_{n+1} = l$.

According to the approach proposed in reference [5], it is possible to divide the entire beam into $(n+1)$ beams connected by massless springs representing the n cracks. As a sequence, the equation of harmonic transverse oscillations of each beam, assumed with uniform cross-section, is:

$$EIy_j^{''''}(x) = \omega^2 \rho S y_j(x) \qquad j = 1, \ldots, n+1, \quad x_{j-1} < x < x_j, \tag{1}$$

where E is the Young's modulus, I is the moment of inertia of the cross-section, ρ is the material density, S is the cross-sectional area of the beam, ω is a natural circular frequency.

It is possible to introduce for each connection between two beams conditions which impose continuity for displacement, bending moment and shear, respectively; moreover, a last equation introduces a discontinuity into the rotation of the beam axis, by imposing equilibrium between transmitted bending moment and rotation of the spring representing the crack [5].

$$\begin{gathered} y_j(x_j) = y_{j+1}(x_j), \\ y_j^{''}(x_j) = y_{j+1}^{''}(x_j), \\ y_j^{'''}(x_j) = y_{j+1}^{'''}(x_j), \\ y_{j+1}^{'}(x_j) - y_j^{'}(x_j) = \Delta_j = c_j y_j^{''}(x_j) \quad j = 1, 2, \ldots, n, \end{gathered} \tag{2}$$

where c_j are the flexibilities of the rotational springs which are functions of the crack extent and beam width. According to reference [5], c_j for one-sided cracks can be expressed as:

$$c_j = 5{\cdot}346 \cdot h \cdot f(\xi_j), \tag{3}$$

where h is the height of the cross-section of the beam, $\xi_j = a_j/h$, where a_j is the depth of the jth crack and

$$\begin{aligned} f(\xi) = 1{\cdot}8624\xi^2 - 3{\cdot}95\xi^3 + 16{\cdot}375\xi^4 - 37{\cdot}226\xi^5 + 76{\cdot}81\xi^6 \\ - 126{\cdot}9\xi^7 + 172\xi^8 - 143{\cdot}97\xi^9 + 66{\cdot}56\xi^{10}. \end{aligned}$$

The case of two-sided cracks can be considered similarly.

The amplitude of displacement $y_j(x)$ can be collected into the function $y(x)$ as follows:

$$y(x) = y_j(x) \quad j = 1, \ldots, n+1, \quad x_{j-1} < x < x_j,$$

such that $y(x)$ allows one to refer to the displacements of the entire beam axis.

Equations (1) with conditions (2) can be expressed through function $y(x)$ as follows:

$$y''''(x) = \lambda^4 y(x) + \sum_{j=1}^{n} \Delta_j \delta''(x - x_j), \tag{4}$$

in which $\delta(x)$ is Dirac's delta function and $\lambda^4 = \omega^2 \rho S/(EI)$. $\delta''(x)$ appears in equation (4) due to the discontinuity in the first derivative of $y(x)$ at the cracks, as expressed in (2).

Furthermore, $y(x)$ is not a smooth function on the interval $[0, l]$ at $x = x_j$. It is possible to introduce a smooth function $y_0(x)$ such as:

$$y(x) = y_0(x) + \sum_{j=1}^{n} \frac{\Delta_j}{2} |x - x_j|. \tag{5}$$

By introducing equation (5) into equation (4), and recalling that:

$$\delta''(x - x_j) = \tfrac{1}{2}(|x - x_j|)'''',$$

the following equation holds:

$$y_0''''(x) = \lambda^4 y_0(x) + \frac{\lambda^4}{2} \sum_{j=1}^{n} \Delta_j |x - x_j|. \tag{6}$$

The general solution of equation (6) can be written in the following way:

$$\begin{aligned} y_0(x) = {} & A\cos(\lambda x) + B\sin(\lambda x) + C\cosh(\lambda x) + D\sinh(\lambda x) \\ & + \frac{\lambda}{4} \sum_{j=1}^{n} \Delta_j \int_0^x (\sinh(\lambda(x - s)) - \sin(\lambda(x - s)))|s - x_j| \, \mathrm{d}s, \end{aligned} \tag{7}$$

where A, B, C and D are constants. By differentiating the previous function twice it is possible to obtain the expression for $y_0''(x)$ at the cracks positions x_i:

$$\begin{aligned} y_0''(x_i) = {} & -A\lambda^2 \cos(\lambda x_i) - B\lambda^2 \sin(\lambda x_i) + C\lambda^2 \cosh(\lambda x_i) \\ & + D\lambda^2 \sinh(\lambda x_i) + \frac{\lambda^3}{4} \sum_{j=1}^{n} \Delta_j M_{ij}(\lambda), \end{aligned} \tag{8}$$

where

$$M_{ij}(\lambda) = \int_0^{x_i} (\sinh(\lambda(x_i - s)) + \sin(\lambda(x_i - s)))|s - x_j| \, \mathrm{d}s. \tag{9}$$

Using equations (8), (9) and the property that

$$\lim_{x \to x_i} y''(x) = y_0''(x_i) \quad i = 1, 2, \ldots, n, \tag{10}$$

the last of conditions (2) can be expressed as:

$$\Delta_i = -Ac_i\lambda^2\cos(\lambda x_i) - Bc_i\lambda^2\sin(\lambda x_i) + Cc_i\lambda^2\cosh(\lambda x_i) + Dc_i\lambda^2\sinh(\lambda x_i) + \frac{c_i\lambda^3}{4}\sum_{j=1}^{n}\Delta_j M_{ij}(\lambda) \quad i = 1, 2, \ldots, n. \tag{11}$$

It is necessary to highlight that equations (11) are valid for all kinds of end conditions of the beam under analysis. Furthermore, equations (11) are a system of n linear equations with $(n+4)$ unknowns (constants A, B, C, D and Δ_i). In order to solve the system it is necessary to introduce four other equations, which are simply obtained by taking into account the end conditions for the beam under analysis. Even though here a cantilever beam is considered, other kinds of end conditions can be analysed similarly.

It is well-known that for a clamped–free beam the end conditions are as follows:

$$y(0) = y'(0) = 0, \quad y''(l) = y'''(l) = 0.$$

Because of $y_0''(l) = y_0''(l)$ and $y'''(l) = y_0'''(l)$, boundary conditions at $x = l$ can be written simply as $y_0''(l) = y_0'''(l) = 0$. Calculating $y_0''(l)$ and $y_0'''(l)$ from equation (7) and taking into account the end conditions, after some reductions it is possible to obtain:

$$-A\cos(\lambda l) - B\sin(\lambda l) + C\cosh(\lambda l) + D\sinh(\lambda l) + \frac{\lambda}{4}\sum_{j=1}^{n}\Delta_j F_j(\lambda) = 0, \tag{12}$$

$$A\sin(\lambda l) - B\cos(\lambda l) + C\sinh(\lambda l) + D\cosh(\lambda l) + \frac{\lambda}{4}\sum_{j=1}^{n}\Delta_j G_j(\lambda) = 0, \tag{13}$$

where

$$F_j(\lambda) = \int_0^l (\sinh(\lambda(l-s)) + \sin(\lambda(l-s)))\,|s - x_j|\,\mathrm{d}s, \tag{14}$$

$$G_j(\lambda) = \int_0^l (\cosh(\lambda(l-s)) + \cos(\lambda(l-s)))\,|s - x_j|\,\mathrm{d}s. \tag{15}$$

In order to write the equations related to the conditions at the clamped end, it is possible to write $y(x)$ from equations (5) and (7) as:

$$y(x) = A\cos(\lambda x) + B\sin(\lambda x) + C\cosh(\lambda x) + D\sinh(\lambda x) + \frac{\lambda}{4}\sum_{j=1}^{n}\Delta_j\int_0^x (\sinh(\lambda(x-s)) - \sin(\lambda(x-s)))\,|s - x_j|\,\mathrm{d}s + \sum_{j=1}^{n}\frac{\Delta_j}{2}|x - x_j|, \tag{16}$$

such as the end condition $y(0) = 0$ becomes

$$A + C + \frac{1}{2}\sum_{j=1}^{n} \Delta_j x_j = 0, \tag{17}$$

while $y'(0) = 0$ is

$$\lambda B + \lambda D - \frac{1}{2}\sum_{j=1}^{n} \Delta_j = 0. \tag{18}$$

Boundary conditions (12), (13), (17) and (18) allow the system of equations (11) to be completed, such that there are $(n + 4)$ linear equations with $(n + 4)$ unknowns. Moreover, the system can be written as:

$$[\mathcal{M}(\lambda)]\{\mathbf{X}\} = \{\mathbf{0}\}.$$

Non-trivial solutions can be obtained by determining values of λ for which the determinant of matrix $[\mathcal{M}(\lambda)]$ vanishes; these values correspond to the eigenfrequencies of the cracked beam. As a consequence, it is evident that by dealing with a matrix with $(n + 4)$ rows and columns instead of $(4n + 4)$ as follows from the extension of the method proposed in reference [5], this procedure will be extremely useful, particularly if used to solve an inverse problem which takes advantage of advanced optimisation techniques, e.g., genetic algorithms and simulated annealing.

The system given by the equations previously listed can be simplified further by expressing the constants C and D via A and B, and Δ_j by using equations (17) and (18) and introducing these expressions in equations (11)–(13):

$$(\cos(\lambda l) + \cosh(\lambda l))A + (\sin(\lambda l) + \sinh(\lambda l))B - \frac{1}{4}\sum_{j=1}^{n}\left(\lambda F_j(\lambda) + \frac{2}{\lambda}\sinh(\lambda l) - 2x_j\cosh(\lambda l)\right)\Delta_j = 0, \tag{19}$$

$$(\sin(\lambda l) - \sinh(\lambda l))A - (\cos(\lambda l) + \cosh(\lambda l))B + \frac{1}{4}\sum_{j=1}^{n}\left(\lambda G_j(\lambda) + \frac{2}{\lambda}\cosh(\lambda l) - 2x_j\sinh(\lambda l)\right)\Delta_j = 0, \tag{20}$$

$$c_i\lambda^2(\cos(\lambda x_i) + \cosh(\lambda x_i))A + c_i\lambda^2(\sin(\lambda x_i) + \sinh(\lambda x_i))B + \sum_{j=1}^{n}\left(\delta_{ij} - \frac{c_i\lambda^3}{4}M_{ij}(\lambda) + \frac{c_i\lambda^2}{2}\cosh(\lambda x_i)x_j - \frac{c_i\lambda}{2}\sinh(\lambda x_i)\right)\Delta_j = 0, \tag{21}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

By introducing the analytical expression for $M_{ij}(\lambda)$, $F_j(\lambda)$ and $G_j(\lambda)$, which can be found in the Appendix, into the previous $n+2$ equations (19)–(21), it is possible to simplify the system as follows:

$$\begin{aligned}(\cos(\lambda l) + \cosh(\lambda l))A + (\sin(\lambda l) + \sinh(\lambda l))B \\ + \frac{1}{4}\sum_{j=1}^{n}\left[x_j(\cos(\lambda l) + \cosh(\lambda l)) - \frac{1}{\lambda}(\sin(\lambda l) + \sinh(\lambda l) + 2\sinh(\lambda(l - x_j)) \right. \\ \left. - 2\sin(\lambda(l - x_j)))\right]\Delta_j = 0,\end{aligned} \tag{22}$$

$$\begin{aligned}(\sin(\lambda l) - \sinh(\lambda l))A - (\cos(\lambda l) + \cosh(\lambda l))B \\ + \frac{1}{4}\sum_{j=1}^{n}\left[x_j(\sin(\lambda l) - \sinh(\lambda l)) + \frac{1}{\lambda}(\cos(\lambda l) + \cosh(\lambda l) + 2\cosh(\lambda(l - x_j)) \right. \\ \left. - 2\cos(\lambda(l - x_j)))\right]\Delta_j = 0,\end{aligned} \tag{23}$$

$$\begin{aligned}c_i\,\lambda^2(\cos(\lambda x_i) + \cosh(\lambda x_i))A + c_i\,\lambda^2(\sin(\lambda x_i) + \sinh(\lambda x_i))B \\ + \sum_{j=1}^{n}(\delta_{ij} + c_i\,R_{ij}(\lambda))\Delta_j = 0, \quad i = 1, \ldots, n,\end{aligned} \tag{24}$$

where

$$R_{ij}(\lambda) = \frac{\lambda^2 x_j}{4}(\cos(\lambda x_i) + \cosh(\lambda x_i)) - \frac{\lambda}{4}(\sin(\lambda x_i) + \sinh(\lambda x_i)), \quad \text{when} \quad i \leqslant j,$$

$$\begin{aligned}R_{ij}(\lambda) = \frac{\lambda^2 x_j}{4}(\cos(\lambda x_i) + \cosh(\lambda x_i)) - \frac{\lambda}{4}(\sin(\lambda x_i) + \sinh(\lambda x_i)) \\ - \frac{\lambda}{2}(\sinh(\lambda(x_i - x_j)) - \sin(\lambda(x_i - x_j))), \quad \text{when} \quad i \leqslant j.\end{aligned}$$

The value of the determinant of a matrix does not change by adding to some column a linear combination of other columns, such that the determinant of the system of linear equations (22)–(24) is equal to the determinant of the following matrix $[\mathbf{U}(\lambda)]$:

$$U_{1,1}(\lambda) = \cos(\lambda l) + \cosh(\lambda l),$$
$$U_{1,2}(\lambda) = \sin(\lambda l) + \sinh(\lambda l),$$
$$U_{1,j+2}(\lambda) = \frac{1}{2\lambda}(\sin(\lambda(l - x_j)) - \sinh(\lambda(l - x_j))),$$
$$U_{2,1}(\lambda) = \sin(\lambda l) - \sinh(\lambda l),$$
$$U_{2,2}(\lambda) = -\cos(\lambda l) - \cosh(\lambda l),$$
$$U_{2,j+2}(\lambda) = \frac{1}{2\lambda}(\cosh(\lambda(l - x_j)) - \cos(\lambda(l - x_j))),$$
$$U_{i+2,1}(\lambda) = c_i\lambda^2(\cos(\lambda x_i) + \cosh(\lambda x_i)),$$
$$U_{i+2,2}(\lambda) = c_i\lambda^2(\sin(\lambda x_i) + \sinh(\lambda x_i)),$$
$$U_{i+2,j+2}(\lambda) = \begin{cases} 0 & i < j, \quad i, j = 1, \ldots, n, \\ \dfrac{c_i\lambda}{2}(\sin(\lambda(x_i - x_j)) - \sinh(\lambda(x_i - x_j))) & i > j, \\ 1 & i = j. \end{cases}$$

Natural frequencies of the cracked beam can be evaluated by solving the non-linear equation

$$\det([\mathbf{U}(\lambda)]) = 0$$

and recalling that it has been assumed that

$$\omega = \lambda^2\sqrt{\frac{EI}{\rho S}}.$$

3. NUMERICAL RESULTS

In order to validate the procedure proposed in this article, results obtained according to this method are compared with available results for beams with one and two cracks.

The evaluation of natural frequencies reduction for a cantilever beam with a crack at the clamped end was addressed by Chondros and Dimarogonas [8]. Using the method proposed in this article it is possible to deal with the problem addressed in reference [8] by locating the crack very close to the clamped end of the beam. In Figure 1 the normalised stiffness of the spring representing the crack (l/c_1) is plotted versus the reduction in the first two natural frequencies, ω_i/ω_{0i} where ω_i and ω_{0i} are natural circular frequencies of the cracked and uncracked beam, respectively. Results obtained in reference [8] are shown, demonstrating a good agreement with that obtained by using the procedure proposed in this article.

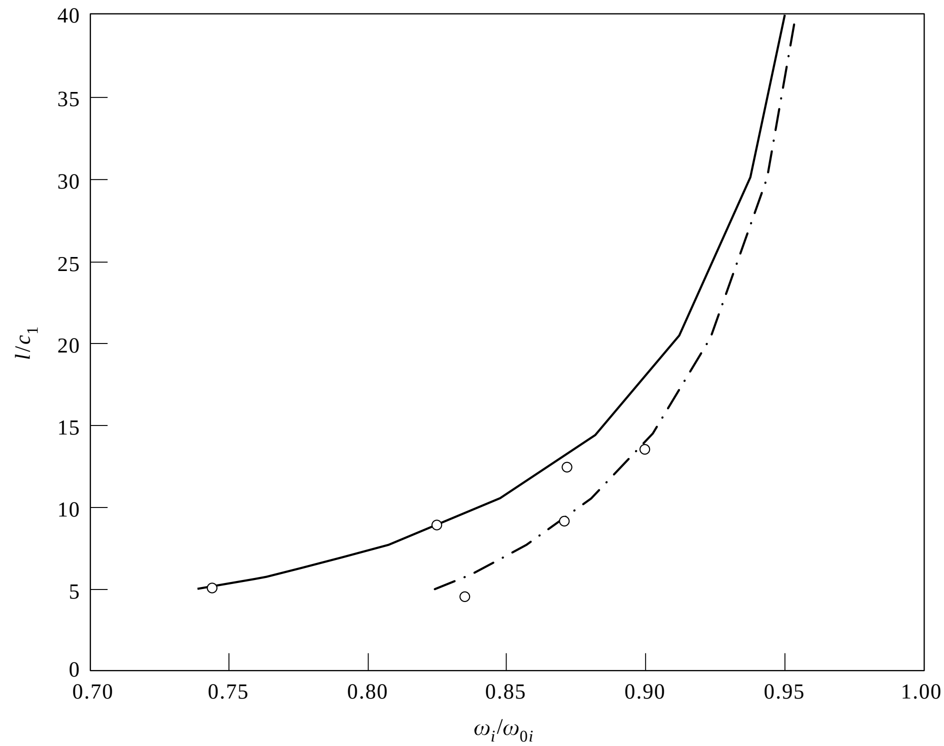

Figure 1. Effect of a single crack at clamped end on the first two natural frequencies (—, ω_1/ω_{01}; $-\cdot-$, ω_2/ω_{02}; ○, reference [8]).

The calculation of natural frequencies for a beam with two cracks was dealt with by Ruotolo *et al.* in reference [7]. The beam under analysis has the following properties: length $l = 0{\cdot}8$ m, rectangular cross-section with width $b = 0{\cdot}02$ m and height $h = 0{\cdot}02$ m, a first crack with position $x_1 = 0{\cdot}12$ m and depth $a_1 = 2$ mm, a second crack with variable position from the clamped to the free end and a depth of 2, 4 and 6 mm. The ratio between the first three natural frequencies of cracked and uncracked beam is shown in Figures 2–4. It is possible to observe that the results are quite close to those published in reference [7] and obtained by using the so-called "continuous model".

Figures 5–7 show the effect of the third crack upon the first three natural frequencies of the cantilever beam with length $l = 0{\cdot}8$ m and rectangular cross-section with width $b = 0{\cdot}02$ m and height $h = 0{\cdot}02$ m. It is assumed that: $a_1 = 6$ mm, $x_1 = 0{\cdot}04$ m, $a_2 = 4$ mm and $x_2 = 0{\cdot}08$ m. The position of the third crack ranges from 0·1 to 0·8 m, and extents of 2, 4 and 6 mm are considered.

In order to show that the reduction in the size of the determinant involved in the natural frequency evaluation has the effect of reducing calculation times with respect to the procedures proposed in references [5] and [6], some comparisons are carried out. The cantilever beam under analysis has the following properties: length $l = 0{\cdot}8$ m, rectangular cross-section with width $b = 0{\cdot}02$ m and height $h = 0{\cdot}02$ m, Young's modulus $E = 2{\cdot}1 \times 10^{11}$ N/m^2, material density $\rho = 7800$ kg/m^3. Results of a first comparison of calculation times related to a

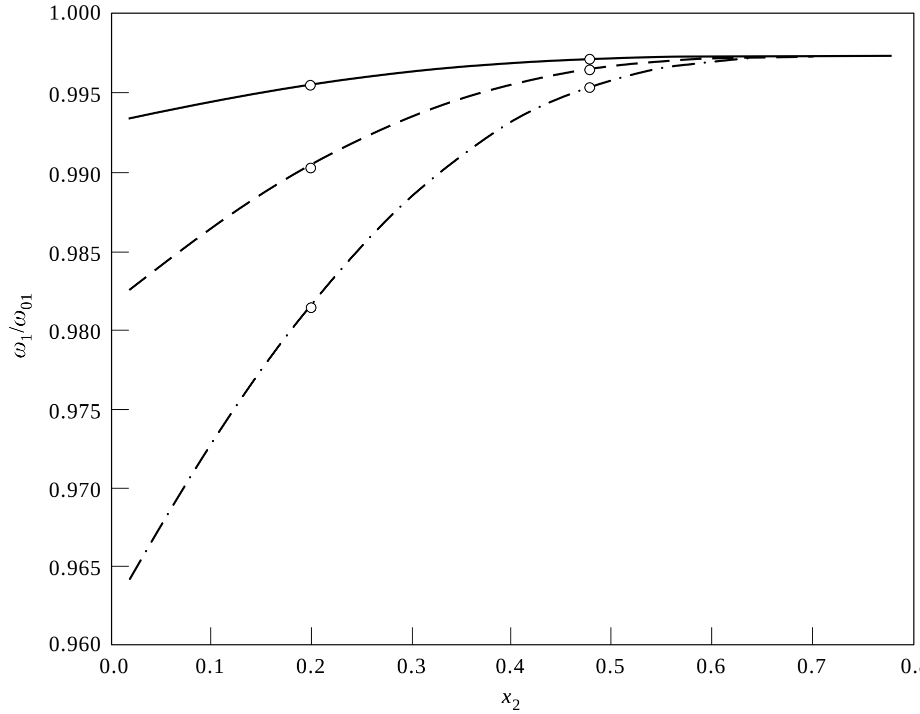

Figure 2. Effect of the second crack on the first natural frequency (—, $a_2/h = 0{\cdot}1$, – –, $a_2/h = 0{\cdot}2$; – · –, $a_2/h = 0{\cdot}3$; ○, reference [7]).

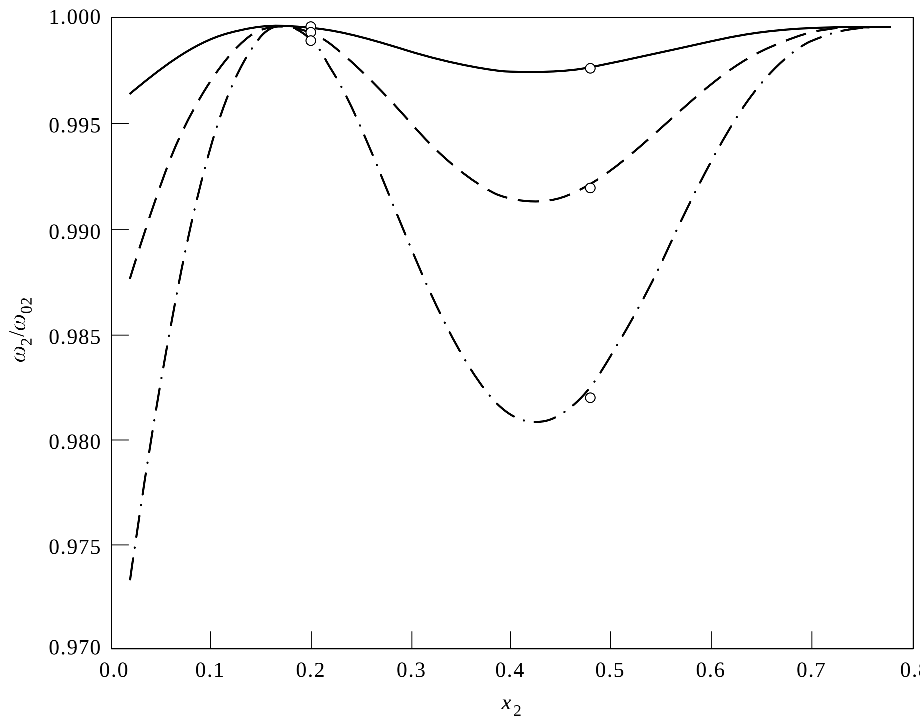

Figure 3. Effect of the second crack on the second natural frequency (—, $a_2/h = 0{\cdot}1$, – –, $a_2/h = 0{\cdot}2$; – · –, $a_2/h = 0{\cdot}3$; ○, reference [7]).

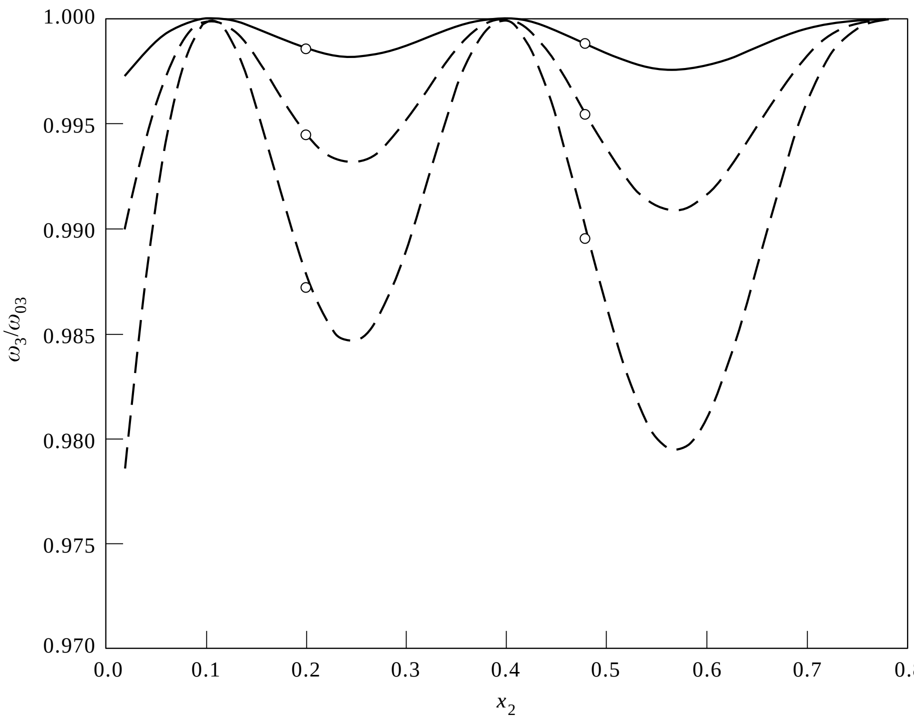

Figure 4. Effect of the second crack on the third natural frequency (—, $a_2/h = 0{\cdot}1$, – –, $a_2/h = 0{\cdot}2$; – · –, $a_2/h = 0{\cdot}3$; ○, reference [7]).

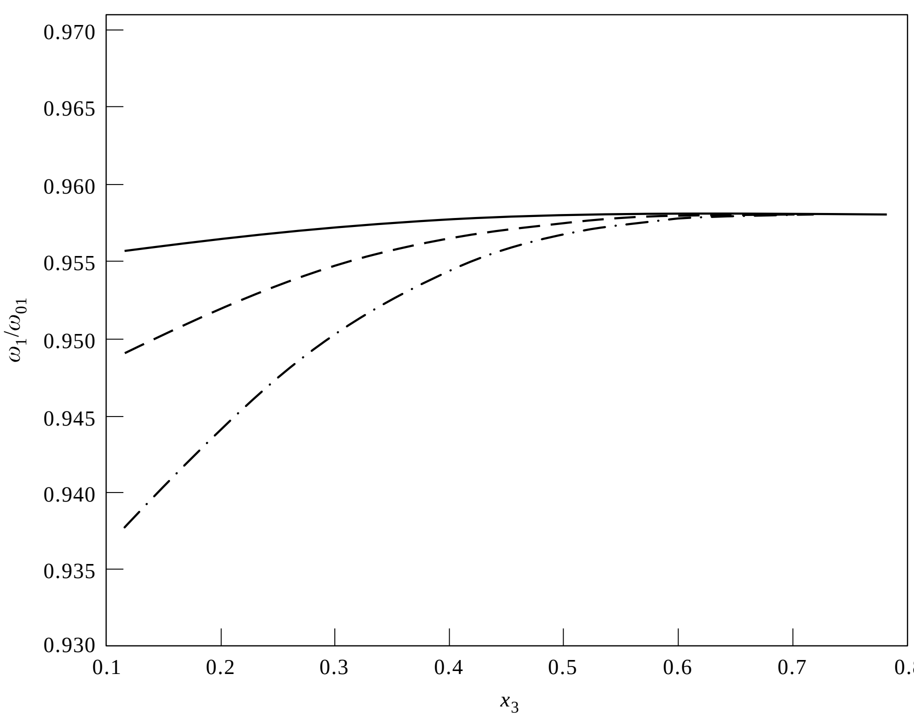

Figure 5. Effect of the third crack on the first natural frequency (—, $a_3/h = 0{\cdot}1$, – –, a_3/h $0{\cdot}2$; – · –, $a_3/h = 0{\cdot}3$).

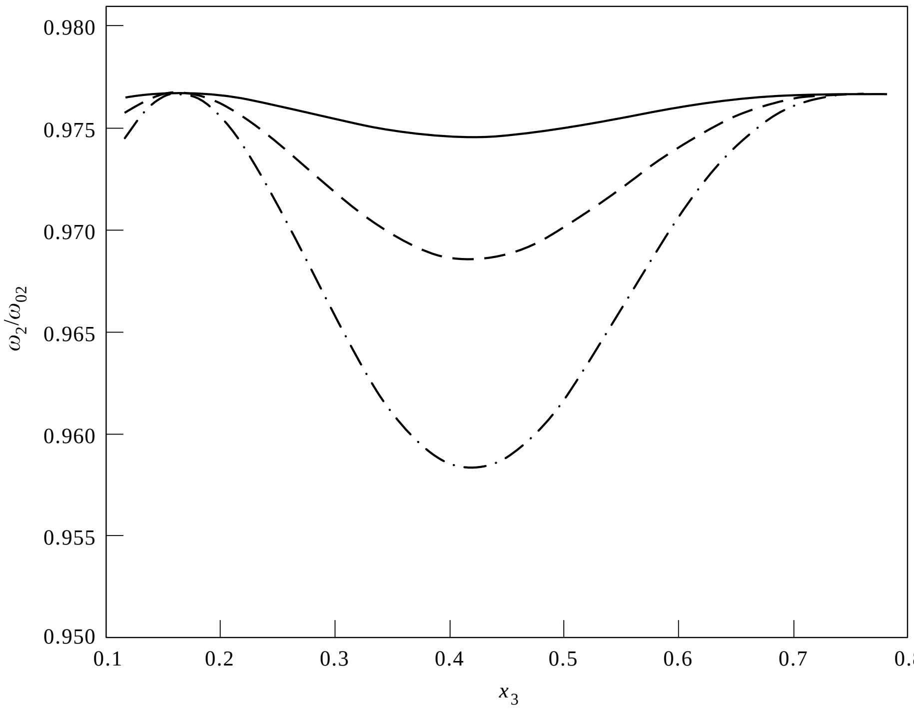

Figure 6. Effect of the third crack on the second natural frequency (—, $a_3/h = 0{\cdot}1$, – –, $a_3/h = 0{\cdot}2$; –·–, $a_3/h = 0{\cdot}3$).

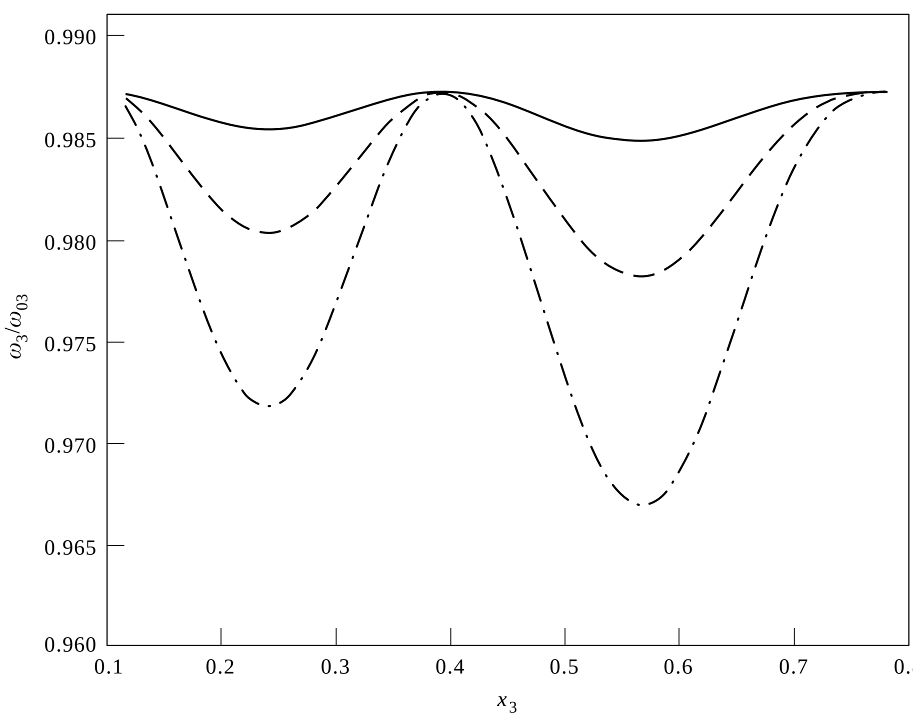

Figure 7. Effect of the third crack on the third natural frequency (—, $a_3/h = 0{\cdot}1$, – –, $a_3/h = 0{\cdot}2$; –·–, $a_3/h = 0{\cdot}3$).

TABLE 1
Comparison of calculation times for a beam with one crack

Natural frequency (Hz)	Starting point (Hz)	Time (present work) (s)	Time (s) (reference [5])
$f_1 = 26{\cdot}1231$	25	0·33	0·96
$f_2 = 164{\cdot}0921$	150	0·50	1·25
$f_3 = 459{\cdot}6028$	450	0·45	0·87

single crack with depth $a_1 = 2$ mm and located at $x_1 = 0{\cdot}12$ m from the clamped end, are shown in Table 1. A similar comparison is performed also for the previous beam with two cracks: the first with depth $a_1 = 2$ mm and located at $x_1 = 0{\cdot}12$ m and the second with depth $a_2 = 3$ mm and position $x_2 = 0{\cdot}4$ m. Related results are shown in Table 2.

Both calculations are performed for the first three natural frequencies, moreover roots of the non-linear equation

$$\det([\mathbf{U}(\lambda)]) = 0$$

were obtained by using the Matlab function fzero(·), which requires a starting point indicated in the tables.

4. CONCLUSIONS

In this paper, as in references [5] and [6], natural frequencies of a cracked beam are evaluated by representing cracks as massless springs and using a continuous mathematical model of the beam in transverse vibration. Utilising this approach, it is possible to write a determinantal equation whose roots are the eigenfrequencies of the beam.

This article demonstrates mathematically that the determinantal equation can be written in a very simple way for any number of cracks. In particular, it is shown that just $(n + 2)$ equations are sufficient to solve the problem for a beam with n cracks, while by extending the procedure proposed in references [5] and [6] to the case of several cracks, $(4n + 4)$ equations are required.

Therefore, a key feature of the procedure is related to the relatively small dimension of the determinant to be evaluated, which enables the times of computation to be reduced. As a consequence, this procedure opens new

TABLE 2
Comparison of calculation times for a beam with two cracks

Natural frequency (Hz)	Starting point (Hz)	Time (present work) (s)	Time (s) (reference [6])
$f_1 = 26{\cdot}0954$	25	0·53	1·56
$f_2 = 163{\cdot}3221$	150	0·71	2·22
$f_3 = 459{\cdot}6011$	450	0·62	1·15

possibilities in the reduction of times needed for solving the inverse problem through advanced optimisation techniques.

ACKNOWLEDGMENTS

The research described in this article was partially funded by the NATO-CNR Fellowship Research Program 1996. The authors would like to thank Professor G. Surace for the opportunity of working in his research group at the Department of Aeronautical and Space Engineering at the Politecnico di Torino and acknowledge useful discussions with Dr C. Surace, Dr D. M. Storer and Dr C. Mares.

REFERENCES

1. S. W. DOEBLING, C. R. FARRAR, M. B. PRIME and D. W. SHEVITZ 1996 *Report no. LA*-13070-*MS*, Los Alamos National Laboratory. Damage identification and health monitoring of structural and mechanical systems from changes in their vibration characteristics: a literature review.
2. A. D. DIMAROGONAS 1996 *Engineering Fracture Mechanics* **55**, 831–857. Vibration of cracked structures: a state of the art review.
3. M. M. YUEN 1985 *Journal of Sound and Vibration* **103**, 301–310. A numerical study of the eigenparameters of a damaged cantilever.
4. A. JOSHI and B. S. MADHUSUDHAN 1991 *Journal of Sound and Vibration* **147**, 475–488. A unified approach to free vibration of locally damaged beams having various homogeneous boundary conditions.
5. P. F. RIZOS, N. ASPRAGATOS and A. D. DIMAROGONAS 1990 *Journal of Sound and Vibration* **138**, 381–388. Identification of crack location and magnitude in a cantilever beam from the vibration modes.
6. W. M. OSTACHOWICZ and M. KRAWCZUK 1991 *Journal of Sound and Vibration* **150**, 191–201. Analysis of the effect of cracks on the natural frequencies of a cantilever beam.
7. R. RUOTOLO, C. SURACE and C. MARES 1996 *Proceedings of* 14*th International Modal Analysis Conference*, 1560–1564. Theoretical and experimental study of the dynamic behaviour of a double-cracked beam.
8. T. G. CHONDROS and A. D. DIMAROGONAS 1980 *Journal of Sound and Vibration* **69**, 531–538. Identification of cracks in welded joints of complex structures.
9. R. Y. LIANG, J. HU and F. CHOY 1992 *Journal of Engineering Mechanics* **118**, 384–396. Theoretical study of crack-induced eigenfrequency changes on beam structures.
10. R. Y. LIANG, J. HU and F. CHOY 1992 *Journal of Engineering Mechanics* **118**, 1468–1487. Quantitative NDE techniques for assessing damages in beam structures.
11. R. Y. LIANG, F. CHOY and J. HU 1991 *Journal of the Franklin Institute* **328** 505–518. Detection of cracks in beam structures using measurements of natural frequencies.
12. J. HU and R. Y LIANG 1993 *Journal of the Franklin Institute* **330**, 841–853. An integrated approach to detection of cracks using vibration characteristics.
13. G.-L. QIAN, S.-N. GU and J.-S. JIANG 1990 *Journal of Sound and Vibration* **138**, 233–243. The dynamic behaviour and crack detection of a beam with a crack.
14. Y. NARKIS 1993 *Journal of Sound and Vibration* **172**, 549–558. Identification of cracks location in vibrating simply supported beams.
15. A. MORASSI 1993 *Journal of Engineering Mechanics* **119**, 1768–1803. Crack-induced changes in eigenfrequencies of beam structures.

16. P. GUDMUNSON 1983 *Journal of Mechanics and Physics of Solids* **31**, 329–345. The dynamic behaviour of slender structures with cross-sectional cracks.
17. S. CHRISTIDES and A. D. S. BARR 1984 *International Journal of Mechanical Science* **26**, 639–648. One-dimensional theory of cracked Euler–Bernoulli beams.
18. G. GOUNARIS and A. D. DIMAROGONAS 1988 *Computer and Structures* **28**, 309–313. A finite-element of a cracked prismatic beam for a structural analysis.
19. T. Y. KAM and T. Y. LEE 1992 *Engineering Fracture Mechanics* **42**, 381–387. Detection of cracks in structures using modal test data.
20. B. S. HAISTY and W. T. SPRINGER 1988 *Journal of Vibration, Acoustics, Stress and Reliability in Design* **110**, 389–394. A general beam element for use in damage assessment of complex structures.
21. R. RUOTOLO and C. SURACE 1997 *Journal of Sound and Vibration* **206**, 567–588. Damage assessment of multiple cracked beams: numerical results and experimental validation.
22. C. MARES and C. SURACE 1996 *Journal of Sound and Vibration* **195**, 195–215. An application of genetic algorithm to identify damage in elastic structures.
23. R. RUOTOLO, C. SURACE and C. MARES 1997 *Proceedings* 15*th International Modal Analysis Conference*, 954–960. Damage identification using simulated annealing.
24. D. C. ZIMMERMAN and K. C. YAP 1997 *Proceedings* 15*th International Modal Analysis Conference*, 551–557. Evolutionary approach for model refinement.
25. K. C. YAP and D. C. ZIMMERMAN 1998 *Proceedings* 16*th International Modal Analysis Conference*, 165–171. The effect of coding on genetic algorithm based structural damage detection.
26. M. I. FRISWELL, J. E. T. PENNY and G. LINDFIELD 1995 *Proceedings* 13*th International Modal Analysis Conference*, 1640–1645. The location of damage from vibration data using genetic algorithms.

APPENDIX

$$M_{ij}(\lambda) = \frac{x_j}{\lambda}(\cosh(\lambda x_i)) - \cos(\lambda x_i)) + \frac{1}{\lambda^2}(\sin(\lambda x_i) - \sinh(\lambda x_i)), \quad i \leqslant j$$

$$M_{ij}(\lambda) = \frac{x_j}{\lambda}(\cosh(\lambda x_i) - \cos(\lambda x_i)) + \frac{1}{\lambda^2}(\sin(\lambda x_i) - \sinh(\lambda x_i) + 2\sinh(\lambda(x_i - x_j)) - 2\sin(\lambda(x_i - x_j))), \quad i > j$$

$$F_j(\lambda) = \frac{x_j}{\lambda}(\cosh(\lambda l) - \cos(\lambda l)) + \frac{1}{\lambda^2}(\sin(\lambda l) - \sinh(\lambda l) + 2\sinh(\lambda(l - x_j)) - 2\sin(\lambda(l - x_j))),$$

$$G_j(\lambda) = \frac{x_j}{\lambda}(\sinh(\lambda l) + \sin(\lambda l)) + \frac{1}{\lambda^2}(\cos(\lambda l) - \cosh(\lambda l) + 2\cosh(\lambda(l - x_j)) - 2\cos(\lambda(l - x_j))).$$